

Geometric global quantum discord

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Abstract. Geometric quantum discord, proposed by Dakic, Vedral, and Brukner [Phys. Rev. Lett. 105 (2010) 190502], is an important measure for bipartite correlations. In this paper, we generalize it to multipartite states, we call the generalized version geometric global quantum discord (GGQD). We characterize GGQD in different ways, give a lower bound for GGQD, and provide some special states which allow analytical GGQD.

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1. Introduction

Quantum correlation is one of the most striking features in quantum theory. Entanglement is by far the most famous and best studied kind of quantum correlation, and leads to powerful applications [1]. Another kind of quantum correlation, called quantum discord, captures more correlations than entanglement in the sense that separable states may also possess nonzero quantum discord. Quantum discord has been attracted much attention in recent years, due to its theoretical interest to quantum theory, and also due to its potential applications [2]. Up to now, the studies on quantum correlations, like entanglement and quantum discord, are mainly focused on the bipartite case.

Quantifying the multipartite correlations is a fundamental and very intractable question. The direct idea is that we can properly generalize the quantifiers of bipartite correlations to the case of multipartite correlations [3, 4, 5, 6, 7]. Recently, generalizing the quantum discord of bipartite states to multipartite states has been discussed in different ways [8, 9, 10, 11]. As an important measure of bipartite correlations, the geometric quantum discord, proposed in [12], has been extensively studied [2]. In this paper, we generalize the geometric quantum discord to multipartite states.

This paper is organized as follows. In Sec.2, we review the definition of geometric quantum discord for bipartite states. In Sec.3, we give the definition of geometric global quantum discord (GGQD) for multipartite states, and give two equivalent expressions for GGQD. In Sec.4, we provide a lower bound for GGQD by using the high order

singular value decomposition of tensors. In Sec.5, we obtain the analytical expressions of GGQD for three classes of states. Sec.6 is a brief summary.

2. Geometric quantum discord of bipartite states

The original quantum discord was defined for bipartite systems over all projective measurements performing only on one subsystem [13, 14]. That is, the quantum discord (with respect to A) of a bipartite state ρ_{AB} of the composite system AB (we suppose $\dim A = n_A < \infty$, $\dim B = n_B < \infty$) was defined as

$$D_A(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}) + \min_{\Pi_A} \{S(\Pi_A(\rho_{AB})) - S(\Pi_A(\rho_A))\}, \quad (1)$$

In Eq.(1), $S(\cdot)$ is Von Neumann entropy, $\rho_A = \text{tr}_B \rho_{AB}$, Π_A is a projective measurement performing on A , $\Pi_A(\rho_{AB})$ is the abbreviation of $(\Pi_A \otimes I_B)(\rho_{AB})$ without any confusion, here I_B is the identity operator of system B . Note that $\Pi_A[\text{tr}_B(\rho_{AB})] = \text{tr}_B[\Pi_A(\rho_{AB})]$, that is, taking partial trace and performing local projective measurement can exchange the ordering.

It can be proved that

$$D_A(\rho_{AB}) \geq 0, \quad (2)$$

$$D_A(\rho_{AB}) = 0 \iff \rho_{AB} = \sum_{i=1}^{n_A} p_i |i\rangle\langle i| \otimes \rho_i^B, \quad (3)$$

where, $n_A = \dim A$, $\{|i\rangle\}_{i=1}^{n_A}$ is any orthonormal basis of system A , $\{\rho_i^B\}_{i=1}^{n_A}$ are density operators of system B , $p_i \geq 0$, $\sum_{i=1}^{n_A} p_i = 1$.

The original definition of quantum discord in Eq.(1) is hard to calculate, even for 2-qubit case, by far we only know a small class of states which allow analytical expressions [2].

Dakic, Vedral, and Brukner proposed the geometric quantum discord, as [12]

$$D_A^G(\rho_{AB}) = \min_{\sigma_{AB}} \{ \text{tr}[(\rho_{AB} - \sigma_{AB})^2] : D_A(\sigma_{AB}) = 0 \}. \quad (4)$$

Obviously,

$$D_A^G(\rho_{AB}) = 0 \iff D_A(\rho_{AB}) = 0. \quad (5)$$

For many cases $D_A^G(\rho_{AB})$ is more easy to calculate than $D_A(\rho_{AB})$ since $D_A^G(\rho_{AB})$ avoided the complicated entropy function. For instance, $D_A^G(\rho_{AB})$ allows analytical expressions for all 2-qubit states [12], and also for all $2 \times d$ ($2 \leq d < \infty$) states [15].

3. Geometric global quantum discord

In [11], the authors generalized the original definition of quantum discord to multipartite states, called global quantum discord (GQD). Consider an N -partite ($N \geq 2$) system, each subsystem A_k ($1 \leq k \leq N$) corresponds Hilbert space H_k with $\dim H_k = n_k$ (we

suppose $n_k < \infty$). The GQD of an N -partite state $\rho_{A_1 A_2 \dots A_N}$ is defined as (here we use an equivalent expression for GQD [16])

$$D(\rho_{A_1 A_2 \dots A_N}) = \sum_{k=1}^N S(\rho_{A_k}) - S(\rho_{A_1 A_2 \dots A_N}) - \max_{\Pi} \left[\sum_{k=1}^N S(\Pi_{A_k}(\rho_{A_k})) - S(\Pi(\rho_{A_1 A_2 \dots A_N})) \right], \quad (6)$$

where, $\Pi = \Pi_{A_1 A_2 \dots A_N}$ is a locally projective measurement on $A_1 A_2 \dots A_N$.

Similar to Eqs.(2, 3), we have Lemma 1 below.

Lemma 1.

$$D(\rho_{A_1 A_2 \dots A_N}) \geq 0, \quad (7)$$

$$D(\rho_{A_1 A_2 \dots A_N}) = 0 \iff \rho_{A_1 A_2 \dots A_N} = \sum_{i_1 i_2 \dots i_N} p_{i_1 i_2 \dots i_N} |i_1\rangle\langle i_1| \otimes |i_2\rangle\langle i_2| \otimes \dots \otimes |i_N\rangle\langle i_N|. \quad (8)$$

Where, $\{|i_k\rangle\}_{i_k=1}^{n_k}$ is any orthonormal basis of H_k , $k = 1, 2, \dots, N$, $p_{i_1 i_2 \dots i_N} \geq 0$, $\sum_{i_1 i_2 \dots i_N} p_{i_1 i_2 \dots i_N} = 1$.

Proof. Eq.(7) is proved in [11]. Eq.(8) can be proved as follows. Noting that $\Pi_{A_1 A_2 \dots A_N}(\rho_{A_1 A_2 \dots A_N}) = \Pi_{A_1}(\Pi_{A_2 \dots}(\Pi_{A_N}(\rho_{A_1 A_2 \dots A_N})))$, then by Eq.(3) and induction, Eq.(8) can be proved.

With Lemma 1, in the same spirit of defining geometric quantum discord for bipartite states in Eq.(4), we now define the GGQD below.

Definition 1. The GGQD of state $\rho_{A_1 A_2 \dots A_N}$ is defined as

$$D^G(\rho_{A_1 A_2 \dots A_N}) = \min_{\sigma_{A_1 A_2 \dots A_N}} \{ \text{tr}[\rho_{A_1 A_2 \dots A_N} - \sigma_{A_1 A_2 \dots A_N}]^2 : D(\sigma_{A_1 A_2 \dots A_N}) = 0 \}. \quad (9)$$

With this definition, it is obvious that

$$D^G(\rho_{A_1 A_2 \dots A_N}) = 0 \iff D(\rho_{A_1 A_2 \dots A_N}) = 0. \quad (10)$$

In [17], two equivalent expressions for Eq.(4) were given (Theorem 1 and Theorem 2 in [17]), and they are very useful for simplifying the calculation of Eq.(4) and yielding lower bound of Eq.(4) [17, 18, 19]. Inspired by this observation, we now derive the corresponding versions of these two equivalent expressions for GGQD. These are Theorem 1 and Theorem 2 below.

Theorem 1. $D^G(\rho_{A_1 A_2 \dots A_N})$ is defined as in Eq.(9), then

$$\begin{aligned} D^G(\rho_{A_1 A_2 \dots A_N}) &= \min_{\Pi} \{ \text{tr}[\rho_{A_1 A_2 \dots A_N} - \Pi(\rho_{A_1 A_2 \dots A_N})]^2 \} \\ &= \text{tr}[\rho_{A_1 A_2 \dots A_N}]^2 - \max_{\Pi} \{ \text{tr}[\Pi(\rho_{A_1 A_2 \dots A_N})]^2 \}, \end{aligned} \quad (11)$$

where, Π is any locally projective measurement performing on $A_1 A_2 \dots A_N$.

Proof. In Eq.(9), for any $\sigma_{A_1 A_2 \dots A_N}$ satisfying $D(\sigma_{A_1 A_2 \dots A_N}) = 0$, $\sigma_{A_1 A_2 \dots A_N}$ can be expressed in the form

$$\rho_{A_1 A_2 \dots A_N} = \sum_{i_1 i_2 \dots i_N} p_{i_1 i_2 \dots i_N} |i_1\rangle\langle i_1| \otimes |i_2\rangle\langle i_2| \otimes \dots \otimes |i_N\rangle\langle i_N|, \quad (12)$$

where, $\{|i_k\rangle\}_{i_k=1}^{n_k}$ is any orthonormal basis of H_k , $k = 1, 2, \dots, N$. $p_{i_1 i_2 \dots i_N} \geq 0$, $\sum_{i_1 i_2 \dots i_N} p_{i_1 i_2 \dots i_N} = 1$. We now expand $\rho_{A_1 A_2 \dots A_N}$ by the bases $\{|i_k\rangle\}_{i_k=1}^{n_k} = \{|j_k\rangle\}_{j_k=1}^{n_k}$, $k = 1, 2, \dots, N$. Then

$$\rho_{A_1 A_2 \dots A_N} = \sum_{i_1 j_1, i_2 j_2, \dots, i_N j_N} \rho_{i_1 j_1, i_2 j_2, \dots, i_N j_N} |i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2| \otimes \dots \otimes |i_N\rangle\langle j_N|, \quad (13)$$

$$\begin{aligned} \text{tr}[\rho_{A_1 A_2 \dots A_N} - \sigma_{A_1 A_2 \dots A_N}]^2 &= \text{tr}[(\rho_{A_1 A_2 \dots A_N})^2] + \sum_{i_1 i_2 \dots i_N} (p_{i_1 i_2 \dots i_N})^2 \\ &\quad - 2 \sum_{i_1 i_2 \dots i_N} \rho_{i_1 i_1, i_2 i_2, \dots, i_N i_N} p_{i_1 i_2 \dots i_N} \\ &= \text{tr}[(\rho_{A_1 A_2 \dots A_N})^2] + \sum_{i_1 i_2 \dots i_N} (\rho_{i_1 i_1, i_2 i_2, \dots, i_N i_N} - p_{i_1 i_2 \dots i_N})^2 \\ &\quad - \sum_{i_1 i_2 \dots i_N} (\rho_{i_1 i_1, i_2 i_2, \dots, i_N i_N})^2. \end{aligned} \quad (14)$$

Hence, it is simple to see that when $\rho_{i_1 i_1, i_2 i_2, \dots, i_N i_N} = p_{i_1 i_2 \dots i_N}$ for all i_1, i_2, \dots, i_N , Eq.(14) achieves its minimum.

Theorem 2. $D^G(\rho_{A_1 A_2 \dots A_N})$ is defined as in Eq.(9), then

$$\begin{aligned} D^G(\rho_{A_1 A_2 \dots A_N}) &= \sum_{\alpha_1 \alpha_2 \dots \alpha_N} (C_{\alpha_1 \alpha_2 \dots \alpha_N})^2 \\ &\quad - \max_{\Pi} \sum_{i_1 i_2 \dots i_N} \left(\sum_{\alpha_1 \alpha_2 \dots \alpha_N} A_{\alpha_1 i_1} A_{\alpha_2 i_2} \dots A_{\alpha_N i_N} C_{\alpha_1 \alpha_2 \dots \alpha_N} \right)^2, \end{aligned} \quad (15)$$

where, $C_{i_1 i_2 \dots i_N}$ and $A_{\alpha_k i_k}$ are all real numbers, they are specified as follows. For any k , $1 \leq k \leq N$, let $L(H_k)$ be the real Hilbert space consisting of all Hermite operators on H_k , with the inner product $\langle X | X' \rangle = \text{tr}(X X')$ for $X, X' \in L(H_k)$. For all k , for given orthonormal basis $\{X_{\alpha_k}\}_{\alpha_k=1}^{n_k^2}$ of $L(H_k)$ (there indeed exists such a basis, see [20]) and orthonormal basis $\{|i_k\rangle\}_{i_k=1}^{n_k}$ of H_k , $C_{i_1 i_2 \dots i_N}$ and $A_{\alpha_k i_k}$ are determined by

$$\rho_{A_1 A_2 \dots A_N} = \sum_{\alpha_1 \alpha_2 \dots \alpha_N} C_{\alpha_1 \alpha_2 \dots \alpha_N} X_{\alpha_1} \otimes X_{\alpha_2} \otimes \dots \otimes X_{\alpha_N}, \quad (16)$$

$$A_{\alpha_k i_k} = \langle i_k | X_{\alpha_k} | i_k \rangle. \quad (17)$$

Proof. According to Eq.(11), and by Eqs.(16, 17), we have

$$\begin{aligned} D^G(\rho_{A_1 A_2 \dots A_N}) &= \text{tr}[\rho_{A_1 A_2 \dots A_N}]^2 - \max_{\Pi} \{ \text{tr}[\Pi(\rho_{A_1 A_2 \dots A_N})]^2 \} \\ &= \sum_{\alpha_1 \alpha_2 \dots \alpha_N} (C_{\alpha_1 \alpha_2 \dots \alpha_N})^2 - \max_{\Pi} \{ \text{tr} \left[\sum_{i_1 i_2 \dots i_N} \sum_{\alpha_1 \alpha_2 \dots \alpha_N} C_{\alpha_1 \alpha_2 \dots \alpha_N} \langle i_1 | X_{\alpha_1} | i_1 \rangle \langle i_2 | X_{\alpha_2} | i_2 \rangle \right. \\ &\quad \left. \dots \langle i_N | X_{\alpha_N} | i_N \rangle |i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2| \otimes \dots \otimes |i_N\rangle\langle j_N| \right]^2 \} \\ &= \sum_{\alpha_1 \alpha_2 \dots \alpha_N} (C_{\alpha_1 \alpha_2 \dots \alpha_N})^2 - \max_{\Pi} \sum_{i_1 i_2 \dots i_N} \left(\sum_{\alpha_1 \alpha_2 \dots \alpha_N} A_{\alpha_1 i_1} A_{\alpha_2 i_2} \dots A_{\alpha_N i_N} C_{\alpha_1 \alpha_2 \dots \alpha_N} \right)^2. \end{aligned} \quad (18)$$

4. A lower bound of GGQD

With the help of Theorem 2, we now provide a lower bound for GGQD.

If we regard $\rho_{A_1 A_2 \dots A_N}$ as a bipartite state in the partition $\{A_k, A_1 A_2 \dots A_{k-1} A_{k+1} \dots A_N\}$, then the original quantum discord and geometric quantum discord of $\rho_{A_1 A_2 \dots A_N}$ with respect to the subsystem A_k can be defined according to Eq.(1) and Eq.(4), we denote

them by $D_{A_k}(\rho_{A_1 A_2 \dots A_N})$ and $D_{A_k}^G(\rho_{A_1 A_2 \dots A_N})$. Comparing Eq.(3) and Eq.(8), it is easy to find that

$$D^G(\rho_{A_1 A_2 \dots A_N}) = 0 \implies D_{A_k}^G(\rho_{A_1 A_2 \dots A_N}) = 0. \quad (19)$$

Consequently, comparing Eq.(4) and Eq.(9), we get

$$D^G(\rho_{A_1 A_2 \dots A_N}) \geq D_{A_k}^G(\rho_{A_1 A_2 \dots A_N}). \quad (20)$$

To proceed further, we need a mathematical fact, called high order singular value decomposition for tensors. We state it as Lemma 2.

Lemma 2. [21] High order singular value decomposition for tensors. For any tensor $T = \{T_{\beta_1 \beta_2 \dots \beta_N} : \beta_k \in \{1, 2, \dots, m_k\}, k = 1, 2, \dots, N\}$, there exist unitary matrices $U^{(k)} = (U_{\beta_k \gamma_k})$, such that

$$T_{\beta_1 \beta_2 \dots \beta_N} = \sum_{\gamma_1 \gamma_2 \dots \gamma_N} U_{\beta_1 \gamma_1}^{(1)} U_{\beta_2 \gamma_2}^{(2)} \dots U_{\beta_N \gamma_N}^{(N)} \Lambda_{\gamma_1 \gamma_2 \dots \gamma_N}, \quad (21)$$

$$\sum_{\gamma_1 \gamma_2 \dots \gamma_{k-1} \gamma_{k+1} \dots \gamma_N} \Lambda_{\gamma_1 \gamma_2 \dots \gamma_{k-1} \gamma_k \gamma_{k+1} \dots \gamma_N}^* \Lambda_{\gamma_1 \gamma_2 \dots \gamma_{k-1} \varepsilon_k \gamma_{k+1} \dots \gamma_N} = s_{\gamma_k}^{(k)} \delta_{\gamma_k \varepsilon_k}, \quad (22)$$

$$s_1^{(k)} \geq s_2^{(k)} \geq \dots \geq s_{n_k}^{(k)} \geq 0. \quad (23)$$

Combining Lemma 2, Eq.(20) and the lower bound of $D_{A_k}^G(\rho_{A_1 A_2 \dots A_N})$ in [17], we can readily obtain a lower bound of $D^G(\rho_{A_1 A_2 \dots A_N})$.

Theorem 3. $D^G(\rho_{A_1 A_2 \dots A_N})$ is defined as in Eq.(9), then a lower bound of $D^G(\rho_{A_1 A_2 \dots A_N})$ is

$$\text{tr}[(\rho_{A_1 A_2 \dots A_N})^2] - \min\left\{\sum_{\gamma_k=1}^{n_k} s_{\gamma_k}^{(k)} : k = 1, 2, \dots, N\right\}, \quad (24)$$

where $s_{\gamma_k}^{(k)}$ are obtained by Lemma 2 in which let $T = \{C_{\alpha_1 \alpha_2 \dots \alpha_N} : \alpha_k \in \{1, 2, \dots, n_k^2\}, k = 1, 2, \dots, N\}$, $C_{\alpha_1 \alpha_2 \dots \alpha_N}$ are defined in Theorem 2.

Proof. Since $D^G(\rho_{A_1 A_2 \dots A_N})$ and $D_{A_i}^G(\rho_{A_1 A_2 \dots A_N})$ keep invariant under locally unitary transformation, hence the state $\rho_{A_1 A_2 \dots A_N}$ in Eq.(16) and the state

$$\Lambda_{A_1 A_2 \dots A_N} = \sum_{\alpha_1 \alpha_2 \dots \alpha_N} \Lambda_{\alpha_1 \alpha_2 \dots \alpha_N} X_{\alpha_1} \otimes X_{\alpha_2} \otimes \dots \otimes X_{\alpha_N}, \quad (25)$$

have the same GGQD, and

$$D_{A_k}^G(\rho_{A_1 A_2 \dots A_N}) = D_{A_k}^G(\Lambda_{A_1 A_2 \dots A_N}). \quad (26)$$

From Eq.(20), we have

$$D^G(\Lambda_{A_1 A_2 \dots A_N}) \geq D_{A_k}^G(\Lambda_{A_1 A_2 \dots A_N}). \quad (27)$$

From the definition of $D_{A_k}^G(\Lambda_{A_1 A_2 \dots A_N})$, Lemma 2 and Theorem 1 in [17], we have

$$\begin{aligned} D_{A_k}^G(\Lambda_{A_1 A_2 \dots A_N}) &= \text{tr}[(\rho_{A_1 A_2 \dots A_N})^2] - \max_{\Pi_{A_k}} \sum_{i_k} \left(\sum_{\alpha_1 \alpha_2 \dots \alpha_N} A_{\alpha_k i_k} \Lambda_{\alpha_1 \alpha_2 \dots \alpha_N} \right)^2 \\ &= \text{tr}[(\rho_{A_1 A_2 \dots A_N})^2] - \max_{\Pi_{A_k}} \sum_{i_k \alpha_k} A_{\alpha_k i_k}^2 s_{\alpha_k}^{(k)} \\ &\geq \text{tr}[(\rho_{A_1 A_2 \dots A_N})^2] - \sum_{\alpha_k=1}^{n_k} s_{\alpha_k}^{(k)}. \end{aligned} \quad (28)$$

We then attain Theorem 3.

5. Examples

We provide some special states which possess analytical GGQD.

Example 1. For N -qubit ($N \geq 2$) Werner-GHZ state

$$\rho = (1 - \mu) \frac{I^{\otimes N}}{2^N} + \mu |\psi\rangle\langle\psi|, \quad (29)$$

the GGQD of ρ is

$$D^G(\rho) = \mu^2/2. \quad (30)$$

In Eq.(29), I is 2×2 identity operator, $\mu \in [0, 1]$, $|\psi\rangle$ is the N -qubit GHZ state

$$|\psi\rangle = (|00\dots 0\rangle + |11\dots 1\rangle)/\sqrt{2}. \quad (31)$$

Proof. We prove Eq.(30) according to Eq.(11).

$\text{tr}(\rho^2)$ can be directly calculated, that is

$$\text{tr}(\rho^2) = \left(\frac{1-\mu}{2^N} + \mu\right)^2 + (2^N - 1)\left(\frac{1-\mu}{2^N}\right)^2. \quad (32)$$

$\max_{\Pi}\{\text{tr}[\Pi(\rho)]^2\}$ can be obtained by the similar calculations of Theorem 4 In [16], the only difference is that the monotonicity of entropy function under majorization relation (Lemma 4 in [16]) will be replaced by the case of the function

$$f(p_1, p_2, \dots, p_n) = -\sum_{i=1}^n p_i^2. \quad (33)$$

That is, $\max_{\Pi}\{\text{tr}[\Pi(\rho)]^2\}$ can be achieved by the eigenvalues

$$\left\{\frac{1-\mu}{2^N} + \frac{\mu}{2}, \frac{1-\mu}{2^N} + \frac{\mu}{2}, \frac{1-\mu}{2^N}, \frac{1-\mu}{2^N}, \dots, \frac{1-\mu}{2^N}\right\}. \quad (34)$$

Thus

$$\max_{\Pi}\{\text{tr}[\Pi(\rho)]^2\} = 2\left(\frac{1-\mu}{2^N} + \frac{\mu}{2}\right)^2 + (2^N - 2)\left(\frac{1-\mu}{2^N}\right)^2. \quad (35)$$

Combine Eqs.(32, 35), we then proved Eq.(30).

Example 2. For N -qubit state

$$\rho = \frac{1}{2^N}(I^{\otimes N} + c_1\sigma_x^{\otimes N} + c_2\sigma_y^{\otimes N} + c_3\sigma_z^{\otimes N}), \quad (36)$$

the GGQD of ρ is

$$D^G(\rho) = \frac{c_1^2 + c_2^2 + c_3^2 - \max\{c_1^2, c_2^2, c_3^2\}}{2^N}. \quad (37)$$

In Eq.(36), I is the 2×2 identity operator, $\{c_1, c_2, c_3\}$ are real numbers constrained by the condition that the eigenvalues of ρ must lie in $[0, 1]$.

Proof. We prove Eq.(37) by using Eq.(11).

$\text{tr}(\rho^2)$ can be directly found, that is

$$\text{tr}(\rho^2) = \frac{c_1^2 + c_2^2 + c_3^2}{2^N}. \quad (38)$$

$\max_{\Pi}\{\text{tr}[\Pi(\rho)]^2\}$ can again be obtained similarly to Theorem 4 In [16], the only difference is that the monotonicity of entropy function under majorization relation (Lemma 4 in [16]) will be replaced by the case of the function Eq.(33).

Similar reduction shows $\max_{\Pi}\{tr[\Pi(\rho)]^2\}$ can be achieved by $\{\frac{1+c}{2^N}\}$, each of them have multiplicity 2^{N-1} , where $c = \max\{|c_1|, |c_2|, |c_3|\}$. Therefore,

$$\max_{\Pi}\{tr[\Pi(\rho)]^2\} = \frac{1+c^2}{2^N}. \quad (39)$$

Combine Eqs.(38, 39), we then get Eq.(37).

Example 3. N-isotropic state

$$\rho = (1-s)\frac{I^{\otimes N}}{d^N} + s|\phi\rangle\langle\phi|, \quad (40)$$

the GGQD of ρ is

$$D^G(\rho) = s^2(1 - \frac{1}{d}), \quad (41)$$

where, $d = \dim H_1$, $H_1 = H_2 = \dots = H_N$, I is the $d \times d$ identity operator, $s \in [0, 1]$, $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{l=1}^d |ll\dots l\rangle$, $\{|l\rangle\}_{l=1}^d$ is an fixed orthonormal basis of H_1 .

Proof. We prove Eq.(41) according to Eq.(9). For any locally projective measurement Π , which corresponds N orthonormal bases of H_1 , we denote them by $\{|i_k\rangle\}_{i_k=1}^d$, $k = 1, 2, \dots, N$. Let $\{|l\rangle\}_{l=1}^d = \{|m\rangle\}_{m=1}^d$. Then,

$$\Pi(|\phi\rangle\langle\phi|) = \frac{1}{d} \sum_{i_1 i_2 \dots i_N, l m} \langle i_1 | l \rangle \langle m | i_1 \rangle \dots \langle i_N | l \rangle \langle m | i_N \rangle |i_1\rangle\langle i_1| \otimes \dots \otimes |i_N\rangle\langle i_N|. \quad (42)$$

$$\begin{aligned} tr\{[|\phi\rangle\langle\phi| - \Pi(|\phi\rangle\langle\phi|)]^2\} &= 1 - \frac{1}{d^2} \sum_{i_1 i_2 \dots i_N} (\sum_{l m} \langle i_1 | l \rangle \langle m | i_1 \rangle \dots \langle i_N | l \rangle \langle m | i_N \rangle)^2 \\ &\geq 1 - \frac{1}{d^2} \sum_{i_1 i_2 \dots i_N} \sum_{l m} \langle i_1 | l \rangle \langle m | i_1 \rangle \dots \langle i_N | l \rangle \langle m | i_N \rangle = 1 - \frac{1}{d}, \end{aligned} \quad (43)$$

and the minimum can be achieved by taking $\langle i_k | l \rangle = \delta_{i_k, l}$, $\langle m | i_k \rangle = \delta_{m, i_k}$. We then proved Eq.(41).

We make some remarks. For states in Eq.(29) and states in Eq.(36), the GQD can also be analytically obtained [16], we then can compare the GQD and GGQD for these two classes of states. For states in Eq.(36) and states in Eq.(40), when $N = 2$, the GGQD in Eq.(37) and Eq.(41) recover the corresponding results in [22].

We also remark that, from Eq.(37), let the state Eq.(36) undergo a locally phase channel performing on any qubit, similar discussions as in [23] show that GGQD may also manifest the phenomena of sudden transition and freeze.

6. Conclusion

In summary, we generalized the geometric quantum discord of bipartite states to multipartite states, we call it geometric global quantum discord (GGQD). We gave different characterizations of GGQD which provided new insights for calculating GGQD. As demonstrations, we provided a lower bound for GGQD by using the high order singular value decomposition of tensors, and obtained the analytical expressions of GGQD for three classes of multipartite states. We also pointed out that GGQD can also manifest the phenomena of sudden transition and freeze.

Understanding and quantifying the multipartite correlations is a very challenging question, we hope that the GGQD proposed in this paper may provide a useful attempt for this issue.

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